
The American Mathematical Monthly

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/uamm20

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**To cite this article:** Hua-Lin Huang, Shengyuan Ruan, Xiaodan Xu & Yu Ye (2025) Solving Algebraic Equations by Completing Powers, The American Mathematical Monthly, 132:3, 218-236, DOI: 10.1080/00029890.2024.2421145

**To link to this article:** https://doi.org/10.1080/00029890.2024.2421145

Published online: 02 Dec 2024.

# Solving Algebraic Equations by Completing Powers

**Hua-Lin Huang, Shengyuan Ruan, Xiaodan Xu, and Yu Ye**

**Abstract.** We derive the Cardano formula of cubic equations by completing the cube, and provide radical solutions to some algebraic equations of degree greater than 3 by completing powers. The main idea of completing the cube and higher powers arises from David Harrison's center theory of higher degree forms. Elementary criteria and solving algorithms for such algebraic equations are presented, and the computation amounts to solving linear equations and quadratic equations.

**1. INTRODUCTION.** Solving algebraic equations is a key problem throughout the whole history of mathematics. The radical solution to a quadratic equation was found by the Babylonians 1500 BC [1]. Three thousands years later, cubic and quartic equations were solved in terms of radicals by Italian mathematicians. After the works of Paolo Ruffini, Niels Henrik Abel and Evariste Galois, it is common knowledge that a general algebraic equation of degree at least 5 has no radical solution. See [1] for more about the history. Theoretically, an equation is solvable by radicals if and only if its Galois group is solvable. In practice, however, it is very hard to determine whether an equation is solvable by radicals.

This article is motivated by the frequently asked question: can one solve cubic equations by completing the cube? There have been many attempts in the literature. For example, Joseph Kung and Gian-Carlo Rota found in [2] radical solutions to cubic equations by the classical invariant theory of binary forms, where the completion of cubes is by virtue of canonical forms of binary cubics. Recently, in [3] Nolan Wallach found a linear fractional transformation that completes the cube for a cubic by geometric invariant theory. The aforementioned works involve complicated computations. The key approach of the present article is an elementary method of completing powers for higher degree forms based on Harrison's theory of centers [4–7]. This can be applied to find radical solutions of some algebraic equations.

Let $f(x) = a_0x^d + \binom{d}{1}a_1x^{d-1} + \cdots + \binom{d}{d-1}a_{d-1}x + a_d$ be a complex polynomial of degree $d > 2$ and let $F(x, y) = a_0x^d + \binom{d}{1}a_1x^{d-1}y + \cdots + a_dy^d$ be its homogenization. Let $H$ be the Hessian matrix of $F(x, y)$ (defined in Section 2). The center $Z(F) := \{X \in \mathbb{C}^{2\times 2} \mid (HX)^T = HX\}$ is a subalgebra of the full matrix algebra $\mathbb{C}^{2\times 2}$. We will prove in Proposition 2 that $Z(F) \cong \mathbb{C} \times \mathbb{C}$ if and only if $F(x, y) = (\alpha_1x + \beta_1y)^d + (\alpha_2x + \beta_2y)^d$, and if and only if $f(x) = (\alpha_1x + \beta_1)^d + (\alpha_2x + \beta_2)^d$, for some $\alpha_i$ and $\beta_j$ with $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. If this is the case, then $f(x) = 0$ is solvable by radicals and the roots are easily obtained. Moreover, the completion of powers is fully indicated by the center and the computation is elementary involving only solving simple linear and quadratic equations.

The article is organized as follows. In Section 2 we recall Harrison's theory of centers and its application to completing powers of polynomials. Then we apply the center theory to derive the Cardano formula of cubic equations in Section 3, and to

doi.org/10.1080/00029890.2024.2421145
MSC: 12D10, 11E76, 15A69

solve some higher degree algebraic equations by completing powers in Section 4. Section 5 is a short summary, in which we include an approach to solve quartic equations by generalized completing powers.

**2. HARRISON CENTERS AND COMPLETING POWERS.** Throughout the paper, let $d > 2$ be an integer and $\Bbbk$ be a field of characteristic 0 or $> d$. As preparation, we recall the notion of center algebras of homogeneous polynomials and their application to a criterion and an algorithm of completing powers.

Let $f(x_1, x_2, \ldots, x_n) \in \Bbbk[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d$. The center of $f = f(x_1, x_2, \ldots, x_n)$ was introduced by Harrison [4] as follows:

$$Z(f) := \{X \in \Bbbk^{n\times n} \mid (HX)^T = HX\},$$

where

$$H = H_{f,\mathbf{x}} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{1\le i,\ j\le n} = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)^T \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) \cdot f$$

is the Hessian matrix.

**Example 1.** *For $d > 2$, consider the sum of powers $f = x_1^d + x_2^d + \cdots + x_n^d$. Then the Hessian matrix $H$ of $f$ is diagonal with $(i, i)$-entry $d(d-1)x_i^{d-2}$. Thus the $(i, j)$-entry of $HX$ reads $d(d-1)x_i^{d-2}c_{ij}$ for $X = (c_{ij})$. The symmetric condition on $HX$ implies easily that $Z(f)$ is the set of diagonal matrices. Thus $Z(f)$ forms a commutative algebra which is isomorphic to $\Bbbk^n$. Notice that the case $d = 2$ is totally different here. For $f = x_1^2 + x_2^2 + \cdots + x_n^2$, the Hessian is two times the identity matrix $I_n$, and in this case, $Z(f) = \{X \in \Bbbk^{n\times n} \mid X^T = X\}$.*

For simplicity we use $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ to denote the column vector of variables. Let $\mathbf{x} = P\mathbf{y}$ be an invertible linear change of variables $\mathbf{x}$, where $P$ is some invertible square matrix and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$. Then $g(\mathbf{y}) = f(P\mathbf{y}) = f(\mathbf{x})$ is a homogeneous polynomial of degree $d$ in variables $y_1, y_2, \ldots, y_n$. Clearly, by the chain rule of derivative we have

$$(\frac{\partial g}{\partial y_1}, \frac{\partial g}{\partial y_2}, \ldots, \frac{\partial g}{\partial y_n}) = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n})P, \tag{1}$$

where each $\frac{\partial f}{\partial x_i}$ is viewed as a polynomial in $\mathbf{y}$ via $\mathbf{x} = P\mathbf{y}$. Hence $H_{g,\mathbf{y}} = P^T H_{f,\mathbf{x}} P$, again the $\frac{\partial^2 f}{\partial x_i \partial x_j}$'s can be viewed as polynomials in $\mathbf{y}$ in the usual way. Then for any $Y \in \Bbbk^{n\times n}$,

$$\begin{aligned}(H_{g,\mathbf{y}}Y)^T = H_{g,\mathbf{y}}Y &\Longleftrightarrow Y^T P^T H_{f,\mathbf{x}} P = P^T H_{f,\mathbf{x}} P Y \\ &\Longleftrightarrow (PYP^{-1})^T H_{f,\mathbf{x}} = H_{f,\mathbf{x}} PYP^{-1},\end{aligned}$$

and it follows that $Z(f) = PZ(g)P^{-1}$.

A homogeneous polynomial is called *nondegenerate* if no variable can be removed by any invertible linear change of its variables. Explicitly, $f(\mathbf{x}) \in \Bbbk[x_1, x_2, \ldots, x_n]$ is nondegenerate if $g(\mathbf{y}) = f(P\mathbf{y}) \notin \Bbbk[y_1, y_2, \ldots, y_{n-1}]$ for any invertible matrix $P$ and linear change of variables $\mathbf{x} = P\mathbf{y}$. Obviously $g(\mathbf{y}) \in \Bbbk[y_1, y_2, \ldots, y_{n-1}]$ if and only if $\frac{\partial g}{\partial y_n} = 0$, and it follows by (1) that $f$ is nondegenerate if and only if its first-order differentials $\frac{\partial f}{\partial x_i}$ are linearly independent. Moreover, for any homogeneous polynomial $f(\mathbf{x})$, there exists some invertible linear change of variables $\mathbf{x} = P\mathbf{y}$, such that $g(\mathbf{y}) = f(P\mathbf{y}) = h(y_1, y_2, \ldots, y_r)$ for some nondegenerate polynomial $h$ in variables

$y_1, y_2, \ldots, y_r$ with $r \leq n$, and if $r < n$ then we may view $g$ as the sum of $h$ and a zero polynomial in variables $y_{r+1}, \ldots, y_n$. We say that $f(x_1, x_2, \ldots, x_n)$ is *diagonalizable* over $\mathbb{k}$ if there exists an invertible $\mathbb{k}$-linear change of variables $\mathbf{x} = P\mathbf{y}$ such that

$$g(\mathbf{y}) = f(P\mathbf{y}) = \lambda_1 y_1^d + \lambda_2 y_2^d + \cdots + \lambda_r y_r^d,$$

where $\lambda_i \in \mathbb{k}^\times$. Note that $r$ is necessarily less than or equal to $n$ here.

The center turns out to be a very effective invariant for deciding whether a homogeneous polynomial is diagonalizable. The following proposition was essentially obtained in [4], see also [5–7].

**Proposition 2.** *Let $f \in \mathbb{k}[x_1, x_2, \ldots, x_n]$ be a homogeneous polynomial of degree $d > 2$. Then*

*(1) $Z(f)$ is a subalgebra of $\mathbb{k}^{n\times n}$, and $Z(f)$ is commutative if and only if $f$ is nondegenerate.*

*(2) $f$ is nondegenerate and diagonalizable over $\mathbb{k}$ if and only if $Z(f) \cong \mathbb{k}^n$ as algebras.*

*Proof.* For the convenience of the reader, we include a proof. In the following, we will use the fact that a polynomial $h(\mathbf{x})$ with $\frac{\partial h}{\partial x_i} = 0$ for all $i$ must be a constant.

By definition, $A = (a_{ij})_{1\leq i,j\leq n} \in Z(f)$ if and only if

$$\sum_{u=1}^{n} a_{uj} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_u} \cdot f = \sum_{u=1}^{n} a_{ui} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_u} \cdot f \tag{2}$$

for any $1 \leq i, j \leq n$. For brevity we write $\partial_i = \frac{\partial}{\partial x_i}$.

(1) First we will show that $Z(f)$ is an algebra, and it suffices to show that $AB \in Z(f)$ for any $A = (a_{ij})_{1\leq i,j\leq n}$, $B = (b_{ij})_{1\leq i,j\leq n} \in Z(f)$.

For any $1 \leq i, j, k \leq n$, direct calculation with multiple applications of (2) shows that

$$\begin{aligned}
\partial_k(\sum_{u,v} a_{vu} b_{ui} \partial_v \partial_j \cdot f) =& \sum_u b_{ui} \partial_j \sum_v a_{vu} \partial_v \partial_k \cdot f = \sum_u b_{ui} \partial_j \sum_v a_{vk} \partial_v \partial_u \cdot f \\
=& \sum_v a_{vk} \partial_v \sum_u b_{ui} \partial_u \partial_j \cdot f = \sum_v a_{vk} \partial_v \sum_u b_{uj} \partial_u \partial_i \cdot f \\
=& \sum_u b_{uj} \partial_i \sum_v a_{vk} \partial_v \partial_u \cdot f = \sum_u b_{uj} \partial_i \sum_v a_{vu} \partial_v \partial_k \cdot f \\
=& \partial_k(\sum_{u,v} a_{vu} b_{uj} \partial_v \partial_i \cdot f).
\end{aligned}$$

Let $AB = (c_{ij})_{1\leq i,j\leq n}$. Then $c_{ij} = \sum_u a_{iu} b_{uj}$ and the above equalities read as

$$\partial_k(\sum_v c_{vi} \partial_v \partial_j \cdot f) = \partial_k(\sum_v c_{vj} \partial_v \partial_i \cdot f), \quad 1 \leq i, j, k \leq n.$$

By the assumption $d > 2$, the difference $\sum_v c_{vi} \partial_v \partial_j \cdot f - \sum_v c_{vj} \partial_v \partial_i \cdot f$ cannot be a constant other than 0. Therefore $\sum_v c_{vi} \partial_v \partial_j \cdot f = \sum_v c_{vj} \partial_v \partial_i \cdot f$ for any $i, j$, and $AB \in Z(f)$ follows.

Next assume that $f$ is nondegenerate. We will show that $AB = BA$. By a similar argument, for any $1 \le i, j, k \le n$, we have

$$\begin{aligned}\partial_k \partial_j (\sum_{u,v} a_{vu} b_{ui} \partial_v \cdot f) &= \sum_u b_{ui} \partial_j \sum_v a_{vu} \partial_v \partial_k \cdot f = \sum_u b_{ui} \partial_j \sum_v a_{vk} \partial_v \partial_u \cdot f \\ &= \sum_v a_{vk} \partial_v \sum_u b_{ui} \partial_u \partial_j \cdot f = \sum_v a_{vk} \partial_v \sum_u b_{uj} \partial_u \partial_i \cdot f \\ &= \sum_u b_{uj} \partial_u \sum_v a_{vk} \partial_v \partial_i \cdot f = \sum_u b_{uj} \partial_u \sum_v a_{vi} \partial_v \partial_k \cdot f \\ &= \sum_v a_{vi} \partial_k \sum_u b_{uj} \partial_u \partial_v \cdot f = \sum_v a_{vi} \partial_k \sum_u b_{uv} \partial_u \partial_j \cdot f \\ &= \partial_k \partial_j (\sum_{u,v} b_{uv} a_{vi} \partial_u \cdot f) = \partial_k \partial_j (\sum_{u,v} b_{vu} a_{ui} \partial_v \cdot f).\end{aligned}$$

Let $BA = (d_{ij})_{1 \le i,j \le n}$. Then the above equalities imply that

$$\sum_v c_{vi} \partial_v \cdot f = \sum_v d_{vi} \partial_v \cdot f$$

for any $i$, again we use the assumption $d > 2$. Now since $f$ is nondegenerate, $c_{vi} = d_{vi}$ for all $v$ and $i$, that is $AB = BA$.

For the converse part, it suffices to show that $Z(f)$ is noncommutative if $f$ is degenerate. By an invertible linear change of variables, we may assume that $\frac{\partial f}{\partial x_n} = 0$. Then the Hessian matrix $H$ has the form $\begin{pmatrix} H_1 & 0 \\ 0 & 0 \end{pmatrix}$, where $H_1$ is a square matrix of order $n - 1$. It is easy to show that

$$\left\{ \begin{pmatrix} aI_{n-1} & 0 \\ \mathbf{c} & d \end{pmatrix} \mid a, d \in \Bbbk, \mathbf{c} \in \Bbbk^{n-1} \right\}$$

is a noncommutative subalgebra of $Z(f)$.

(2) Suppose $f$ is nondegenerate and diagonalizable. Then there is a change of variable $\mathbf{x} = P\mathbf{y}$ such that

$$g(\mathbf{y}) := f(P\mathbf{y}) = \lambda_1 y_1^d + \lambda_2 y_2^d + \cdots + \lambda_n y_n^d$$

for some $\lambda_i \in \Bbbk^\times$. As in Example 1, it is easy to see that $Z(g)$ consists of all diagonal matrices. Then $Z(f) = PZ(g)P^{-1}$ is isomorphic to $\Bbbk^n$ as algebras.

Conversely, suppose $Z(f) \cong \Bbbk^n$ as algebras. Since $Z(f)$ is commutative, it follows from (1) that $f$ is nondegenerate. Moreover, in $Z(f)$ there exists a complete set of orthogonal primitive idempotents, say nonzero elements $e_1, e_2, \ldots, e_n$ with $1 = e_1 + e_2 + \cdots + e_n$, $e_i^2 = e_i$ for any $i$, and $e_i e_j = 0$ for $i \ne j$. Recall that a nonzero element $e$ is called an *idempotent* if $e^2 = e$, and two idempotents $e$ and $f$ are orthogonal if $ef = fe = 0$.

As the polynomial $x^2 - x$ has no multiple roots, all the $e_i$'s are diagonalizable under conjugation. Since they mutually commute, there exists some invertible matrix $Q \in \Bbbk^{n \times n}$ such that $Q^{-1} e_i Q$ is diagonal for all $i$, see for instance Theorem 7, Section 6.5 in [8].

An easy exercise in linear algebra shows that if two diagonal matrices $A, B$ are orthogonal, i.e., $AB = BA = 0$, then $\operatorname{rank}(A + B) = \operatorname{rank}(A) + \operatorname{rank}(B)$. By induction,

$$\sum_{i=1}^{n} \operatorname{rank}(Q^{-1}e_i Q) = \operatorname{rank}(\sum_{i=1}^{n} Q^{-1}e_i Q) = \operatorname{rank}(I_n) = n,$$

which forces $\operatorname{rank}(e_i) = \operatorname{rank}(Q^{-1}e_i Q) = 1$ for all $i$. Moreover, $e_i^2 = e_i$ implies that $e_i$ has eigenvalue 1 of multiplicity 1.

For each $i$, take an eigenvector $X_i$ of $e_i$ for the eigenvalue 1, and set $P = (X_1, X_2, \dots, X_n)$ to be the matrix whose columns are the $X_i$'s. Then $e_i X_i = X_i$ for any $i$ and $e_i X_j = 0$ for $i \neq j$. It follows that $X_1, X_2, \dots, X_n$ are linearly independent, and hence $P$ is invertible. Clearly $P^{-1}e_i P = E_{ii}$, where $E_{ii}$ is the matrix with $(i, i)$-entry 1 and 0 otherwise.

Now take the change of variables $\mathbf{x} = P\mathbf{y}$, and set $g(\mathbf{y}) = f(P\mathbf{y})$. Then $Z(g) = P^{-1}Z(f)P$ contains $E_{ii}$, $1 \leq i \leq n$. Let $G = (g_{ij})_{1\leq i,j\leq n}$ be the Hessian of $g$. Since $E_{ii} \in Z(g)$, we have $GE_{ii} = E_{ii}^T G = E_{ii}G$. If $i \neq j$, comparing the $(i, j)$-entry on both sides gives $0 \cdot g_{ij} = g_{ij}$. Thus $g_{ij} = 0$ for $i \neq j$, which means that $G$ is a diagonal matrix. It follows easily that $g(y) = \lambda_1 y_1^d + \lambda_2 y_2^d + \dots + \lambda_n y_n^d$, i.e., $f(x)$ is diagonalizable. ∎

**Remark 3.** *The proof of the previous proposition relies heavily on the assumption that $d > 2$. In fact, as we have shown in Example 1, the center of $f = x_1^2 + x_2^2 + \dots + x_n^2$ consists of all symmetric matrices, and hence is NOT closed under multiplication unless $n = 1$.*

**Remark 4.** *The previous proposition provides a criterion for completing powers of multivariate homogeneous polynomials. In addition, the proof also contains an algorithm for the process of completing powers for diagonalizable polynomials, which is summarized as follows:*

1. *Find the Hessian matrix $H$ of a given homogeneous polynomial.*
2. *Solve the linear system of matrix equations $(HX)^T = HX$ to find the center $Z(f)$.*
3. *Verify whether $Z(f)$ is isomorphic to $\mathbb{k}^n$ as algebras, and find a complete set of orthogonal idempotents $e_1, e_2, \dots, e_n$ if it is the case.*
4. *For each $e_i$ find an eigenvector $X_i$ for the eigenvalue 1.*
5. *Set $P = (X_1, X_2, \dots, X_n)$, and take the change of variable $\mathbf{x} = P\mathbf{y}$. Then $g(\mathbf{y}) = f(P\mathbf{y})$ is a diagonal form.*

For the previous item 3, we mention that a commutative subalgebra $Z \subseteq \mathbb{k}^{n\times n}$ is isomorphic to $\mathbb{k}^n$ if and only if $\dim(Z) = n$ and there exists a basis of $Z$ consisting of diagonalizable matrices. Note that a square matrix $A$ is diagonalizable if and only if its minimal polynomial has no multiple roots. The minimal polynomial is obtained as follows. First find the maximal $d$ such that $I_n, A, \cdots, A^{d-1}$ are linearly independent, and then find $a_0, \cdots, a_{d-1}$ such that $A^d = -a_{d-1}A^{d-1} - \cdots - a_0 I_n$. Then $p(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0$ is the minimal polynomial of $A$. Now $p(x)$ has no multiple roots if and only if $p(x)$ is coprime to its derivative $p'(x)$, or equivalently $(p(x), p'(x)) = 1$, where $(p(x), p'(x))$ is the greatest common divisor which is calculated using the famous Euclidean algorithm.

For a commutative subalgebra $Z \subseteq \mathbb{k}^{n\times n}$, finding a complete set of orthogonal idempotents needs a bit of work. In general we need to solve a certain system of quadratic equations. Assume that $Z \cong \mathbb{k}^n$, and $\{e_1, e_2, \ldots, e_n\}$ is a complete set of orthogonal idempotents, that is $1 = e_1 + e_2 + \cdots + e_n$, $e_i^2 = e_i$ for any $i$ and $e_i e_j = 0$ for $i \neq j$. Then, as we have shown above, $\mathrm{tr}(e_i) = \mathrm{rank}(e_i) = 1$ for any $i$. Moreover, any $e \in Z$ with the property $\mathrm{tr}(e) = \mathrm{rank}(e) = 1$ is equal to some $e_i$. Thus to find the desired idempotents we need to find elements in $Z$ whose trace and rank are both 1.

Among the steps of the previous algorithm, finding a complete set of orthogonal idempotents is the most difficult one. This is equivalent to finding a set of matrices with trace 1 and rank 1 which is computationally very expensive. The rank 1 condition can be transformed to a system of quadratic equations. If the number of variables is two, then there is only one quadratic equation which can be easily solved. However if the number of variables is greater than two, then the system of quadratic equations may become very complicated.

We conclude this section with an example to elucidate the algorithm of completing powers.

**Example 5.** *Consider the following ternary cubic*

$$f(x_1, x_2, x_3) = x_1^3 + 3x_2x_1^2 + 3x_3x_1^2 + 3x_2^2x_1 + 3x_3^2x_1 + 6x_2x_3x_1 - x_2^3 + 20x_3^3 - 21x_2x_3^2 + 15x_2^2x_3.$$

*Then the Hessian $H$ of $f$ is*

$$\begin{pmatrix} 6x_1 + 6x_2 + 6x_3 & 6x_1 + 6x_2 + 6x_3 & 6x_1 + 6x_2 + 6x_3 \\ 6x_1 + 6x_2 + 6x_3 & 6x_1 - 6x_2 + 30x_3 & 6x_1 + 30x_2 - 42x_3 \\ 6x_1 + 6x_2 + 6x_3 & 6x_1 + 30x_2 - 42x_3 & 6x_1 - 42x_2 + 120x_3 \end{pmatrix}$$

$$= 6x_1 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + 6x_2 \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 5 \\ 1 & 5 & -7 \end{pmatrix} + 6x_3 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & -7 \\ 1 & -7 & 20 \end{pmatrix}.$$

*Suppose $X = (x_{ij}) \in Z(f)$. By comparing the coefficients of monomials appearing in $H$, the condition $(HX)^T = HX$ can be translated into a system of linear equations in $x_{ij}$, say*

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} X = X^T \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 5 \\ 1 & 5 & -7 \end{pmatrix} X = X^T \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 5 \\ 1 & 5 & -7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & -7 \\ 1 & -7 & 20 \end{pmatrix} X = X^T \begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & -7 \\ 1 & -7 & 20 \end{pmatrix}.$$

*A general solution to the system of linear equations reads*

$$X = \begin{pmatrix} a & a - b & a + 2b - 3c \\ 0 & b & -2b + 2c \\ 0 & 0 & c \end{pmatrix}.$$

*By direct verification, we have*

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

*are the desired idempotents, and* $(1, 0, 0)^T, (-1, 1, 0)^T, (-3, 2, 1)^T$ *are their eigenvectors for the eigenvalue 1 respectively. Now set* $P = \left(\begin{smallmatrix} 1 & -1 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ *and take a change of variables* $\mathbf{x} = P\mathbf{y}$. *Then* $f(P\mathbf{y})$ *has the form* $\lambda_1 y_1^3 + \lambda_2 y_2^3 + \lambda_3 y_3^3$, *and by direct computation we have*

$$f(P\mathbf{y}) = y_1^3 - 2y_2^3 + 3y_3^3.$$

*Note that* $P^{-1} = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, *and* $y_1 = x_1 + x_2 + x_3,\ y_2 = x_2 - 2x_3,\ y_3 = x_3$, *therefore*

$$f(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^3 - 2(x_2 - 2x_3)^3 + 3x_3^3.$$

**3. CARDANO FORMULA REVISITED BY COMPLETING THE CUBE.** In this section, we derive the well-known Cardano formula of cubic equations by completing the cube. Let

$$a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3 = 0 \tag{3}$$

be a general cubic equation over the field $\mathbb{C}$ of complex numbers. Similar to the situation of quadratic equations but going a bit further, we aim to express cubic equations as the sum of two cubes of linear binomials. That is, we try to find an identity as the following

$$f(x) = a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3 = (\alpha_1 x + \beta_1)^3 + (\alpha_2 x + \beta_2)^3. \tag{4}$$

If such an identity is found, then the solution to (3) is easily obtained by taking the cubic roots of both sides of $(\alpha_1 x + \beta_1)^3 = -(\alpha_2 x + \beta_2)^3$.

Thanks to the theory of centers, we have an effective criterion and algorithm of completing the cubes for a cubic equation as (4). In order to apply Proposition 2, we homogenize univariate cubic polynomials as binary cubics

$$F(x, y) = a_0 x^3 + 3a_1 x^2 y + 3a_2 x y^2 + a_3 y^3. \tag{5}$$

We say that $f(x)$ is *nondegenerate* if $F(x, y)$ is nondegenerate. It is clear that

$$f(x) = (\alpha_1 x + \beta_1)^3 + (\alpha_2 x + \beta_2)^3 \Leftrightarrow F(x, y) = (\alpha_1 x + \beta_1 y)^3 + (\alpha_2 x + \beta_2 y)^3. \tag{6}$$

We start with computing the center of $F(x, y)$. The Hessian of $F(x, y)$ is given by

$$H = \begin{pmatrix} 6a_0 x + 6a_1 y & 6a_1 x + 6a_2 y \\ 6a_1 x + 6a_2 y & 6a_2 x + 6a_3 y \end{pmatrix} = 6x \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix} + 6y \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}.$$

Note that $F(x, y)$ is assumed to be nonzero throughout, hence so is the Hessian. Suppose $X = (c_{ij}) \in \mathbb{C}^{2\times 2}$ and $X \in Z(F)$. Then by comparing the coefficients of $x$

and $y$ in the equality $HX = X^T H$, we have

$$\begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{pmatrix} \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix},$$

$$\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}.$$

Equivalently the $c_{ij}$'s satisfy the following linear equations

$$\begin{cases} a_0 c_{12} + a_1(c_{22} - c_{11}) - a_2 c_{21} = 0, \\ a_1 c_{12} + a_2(c_{22} - c_{11}) - a_3 c_{21} = 0. \end{cases} \tag{7}$$

It follows that $Z(F)$ has dimension 2 or 3, and $\dim Z(F) = 3$ if and only if $(a_0, a_1, a_2)$ and $(a_1, a_2, a_3)$ are linearly dependent. An interesting observation is that the latter condition is equivalent to saying that $F$ is degenerate. In fact, $F(x, y)$ is degenerate if and only if

$$\frac{\partial F(x, y)}{\partial x} = 3a_0 x^2 + 6a_1 xy + 3a_2 y^2,$$

$$\frac{\partial F(x, y)}{\partial y} = 3a_1 x^2 + 6a_2 xy + 3a_3 y^2$$

are linearly dependent, if and only if

$$\operatorname{rank} \begin{pmatrix} 3a_0 & 6a_1 & 3a_2 \\ 3a_1 & 6a_2 & 3a_3 \end{pmatrix} = 1,$$

if and only if

$$\operatorname{rank} \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{pmatrix} = 1.$$

We have shown that if $F$ is degenerate, then $Z(F)$ is a 3-dimensional subalgebra of $\mathbb{C}^{2\times 2}$, which is not commutative. If $F$ is nondegenerate, then $Z(F)$ is a 2-dimensional commutative subalgebra of $\mathbb{C}^{2\times 2}$. Note that the commutativity or non-commutativity of the center algebra $Z(F)$ follows from Proposition 2.

It is well known that a 2-dimensional algebra $Z$ over $\mathbb{C}$ is isomorphic either to $\mathbb{C}^2$, or to $\mathbb{C}[\varepsilon]/(\varepsilon^2)$. In fact, take $\alpha \in Z \setminus \mathbb{C}$, then $1, \alpha$ form a basis of $Z$. Hence $\alpha^2 + b\alpha + c = 0$ for some $b, c \in \mathbb{C}$, and $Z \cong \mathbb{C}[x]/(x^2 + bx + c)$ as algebras. There are two cases.

If $b^2 - 4c \neq 0$, then $x^2 + bx + c$ has distinct roots $a_1$ and $a_2$, then we have isomorphisms

$$Z \cong \mathbb{C}[x]/((x - a_1)(x - a_2)) \cong \mathbb{C}[x]/(x - a_1) \times \mathbb{C}[x]/(x - a_2) \cong \mathbb{C} \times \mathbb{C}$$

of algebras, where the second isomorphism comes from the Chinese Remainder Theorem and the fact $x - a_1$ and $x - a_2$ are coprime for $a_1 \neq a_2$. In fact, $e_1 = \frac{\alpha - a_2}{a_1 - a_2}$ and $e_2 = \frac{\alpha - a_1}{a_2 - a_1}$ are orthogonal idempotents of $Z$, and $Z = \mathbb{C}e_1 \times \mathbb{C}e_2 \cong \mathbb{C} \times \mathbb{C}$.

If $b^2 - 4c = 0$, then $x^2 + bx + c = (x-a)^2$ for some $a$, then $Z \cong \mathbb{C}[x]/((x-a)^2) \cong \mathbb{C}[\varepsilon]/(\varepsilon^2)$, the ring of dual numbers.

So far we have shown that for any nonzero binary cubic form $F$, exactly one of the following situations occur: (I) $\dim Z(F) = 3$; (II) $Z(F) \cong \mathbb{C}^2$; (III) $Z(F) \cong \mathbb{C}[\varepsilon]/(\varepsilon^2)$. We will deal with these three cases in Proposition 6, Proposition 8, and Theorem 10 separately.

First we derive the following well-known facts.

**Proposition 6.** *Let $F(x,y) = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3$. Then the following statements are equivalent.*

*(1)* $\dim(Z(F)) = 3$.

*(2)* *$F(x,y)$ is degenerate.*

*(3)* *$(a_0, a_1, a_2)$ and $(a_1, a_2, a_3)$ are linearly dependent.*

*(4)* *$F(x,y)$ is a perfect cube, i.e., $F(x,y) = (\alpha x + \beta y)^3$ for some $\alpha, \beta \in \mathbb{C}$. In this case,*
$$(\alpha, \beta) = \begin{cases} \frac{1}{\sqrt[3]{a_0^2}}(a_0, a_1), & \text{if } \quad a_0 \neq 0; \\ (0, \sqrt[3]{a_3}), & \text{if } \quad a_0 = 0. \end{cases}$$

*Consequently, let $f(x) = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$ be a degenerate cubic polynomial, then $f(x)$ has a unique root $-\frac{a_1}{a_0}$ with multiplicity 3.*

*Proof.* We are left to show that (4) is equivalent to (1)–(3). Assume $F(x,y) = (\alpha x + \beta y)^3$. Then $\frac{\partial F}{\partial x} = 3\alpha(\alpha x + \beta y)^2$ and $\frac{\partial F}{\partial y} = 3\beta(\alpha x + \beta y)^2$ are clearly linearly dependent.

Conversely, assume $(a_0, a_1, a_2)$ and $(a_1, a_2, a_3)$ are linearly dependent. If $a_0 = 0$, then the assumption on the rank forces $a_1 = a_2 = 0$, and hence $F(x,y) = a_3y^3 = (\sqrt[3]{a_3}y)^3$.

If $a_0 \neq 0$, then $(a_1, a_2, a_3) = \frac{a_1}{a_0}(a_0, a_1, a_2)$, hence
$$F(x,y) = a_0\left(x^3 + 3\frac{a_1}{a_0}x^2y + 3\left(\frac{a_1}{a_0}\right)^2xy^2 + \left(\frac{a_1}{a_0}\right)^3y^3\right) = a_0\left(x + \frac{a_1}{a_0}y\right)^3.$$

That is, $F(x,y)$ is a perfect cube in either case.

The last assertion follows easily from (4). Note that $a_0 \neq 0$ in this case because $f$ is a cubic. ■

In the previous proposition, it is shown that a degenerate binary cubic form can be completed as one perfect cube. Next we turn to the nondegenerate case.

Let $F(x,y)$ be a nondegenerate cubic form. Then the coefficient matrix of the linear system (7) has rank 2. For brevity we denote its minors of order 2 by
$$D_1 = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}.$$

Clearly, rank $\left(\begin{smallmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{smallmatrix}\right) = 2$ if and only if at least one of the $D_i$'s is nonzero.

By direct calculation, a general element of $Z(F)$, or equivalently a solution to (7) reads
$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -D_3 \\ D_1 & D_2 \end{pmatrix}, \quad \forall a, b \in \mathbb{C}.$$

Let $\Lambda$ denote the last matrix. Then the minimal polynomial of $\Lambda$ is $x^2 - D_2x + D_1D_3$. By the above discussion, we know that $Z(F)$ is isomorphic to $\mathbb{C}^2$ if $D_2^2 - 4D_1D_3 \neq 0$, otherwise $Z(F)$ is isomorphic to $\mathbb{C}[\varepsilon]/(\varepsilon^2)$.

**Theorem 7.** *Let* $F(x, y) = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3$. *Then the following statements are equivalent.*

(1) $Z(F) \cong \mathbb{C}^2$ *as algebras.*

(2) $F(x, y) = (\alpha_1 x + \beta_1 y)^3 + (\alpha_2 x + \beta_2 y)^3$ *with* $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$.

(3) $D_2^2 - 4D_1D_3 \neq 0$.

*Consequently, a general complex binary cubic is the sum of cubes of two linearly independent linear forms.*

*Proof.* By Proposition 2, $F(x, y) = (\alpha_1 x + \beta_1 y)^3 + (\alpha_2 x + \beta_2 y)^3$ with $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$ if and only if $F$ is nondegenerate and diagonalizable, if and only if $Z(F) \cong \mathbb{C}^2$, if and only if $D_2^2 - 4D_1D_3 \neq 0$.

The preceding assertion implies that the set of binary cubics that are sums of cubes of two different linear forms is a principal open set in the affine space of all binary cubics. Since this set is obviously not empty, it is dense by the theory of elementary algebraic geometry [**9**, Chapter 4]. In other words, a general binary cubic is the sum of cubes of two different linear forms. ∎

Now we complete the cube as a sum of two perfect cubes for a general binary cubic. Keep the notation

$$\Lambda = \begin{pmatrix} 0 & -D_3 \\ D_1 & D_2 \end{pmatrix}, \tag{8}$$

where

$$D_1 = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}.$$

Assume $D_2^2 - 4D_1D_3 \neq 0$. Then $\Lambda$ has two distinct eigenvalues

$$\lambda_{1,2} = \frac{D_2 \pm \sqrt{D_2^2 - 4D_1D_3}}{2}.$$

We stress that by $\sqrt{a}$ we always mean a fixed square root of $a$, and the other one is $-\sqrt{a}$. Similarly, $\sqrt[3]{a}$ refers to a fixed cubic root of $a$, and all cubic roots of $a$ are given by $\omega^s\sqrt[3]{a}$, $s = 0, 1, 2$, where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

If $D_1 = 0$, then $D_2 \neq 0$. In this case, $a_0 \neq 0$. Otherwise assume $a_0 = 0$, $D_2 = -a_1a_2 \neq 0$ forces $a_1, a_2 \neq 0$, and hence $D_1 = -a_1^2 \neq 0$, which leads to a contradiction. In this case, it is straightforward to show that

$$F(x, y) = a_0(x + \frac{a_1}{a_0}y)^3 + (a_3 - \frac{a_1^3}{a_0^2})y^3.$$

Now we assume $D_1 \neq 0$. By computing the eigenvectors of $\Lambda$, we have

$$\begin{pmatrix} 0 & -D_3 \\ D_1 & D_2 \end{pmatrix}\begin{pmatrix} -\lambda_2 & -\lambda_1 \\ D_1 & D_1 \end{pmatrix} = \begin{pmatrix} -\lambda_2 & -\lambda_1 \\ D_1 & D_1 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{9}$$

As in the proof of Proposition 2, take the change of variables

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\lambda_2 & -\lambda_1 \\ D_1 & D_1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{10}$$

then we have

$$F(x, y) = au^3 + bv^3$$

for some $a, b \in \mathbb{C}$. One can determine $a$ and $b$ by comparing the coefficients of both sides of the preceding equation. As a summary of the previous discussion, we have

**Proposition 8.** *Let* $F(x, y) = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3$ *and suppose* $D_2^2 - 4D_1D_3 \neq 0$.

*(1) If* $D_1 = 0$*, then* $a_0 \neq 0$ *and* $F(x, y) = a_0(x + \frac{a_1}{a_0}y)^3 + (a_3 - \frac{a_1^3}{a_0^2})y^3$.

*(2) If* $D_3 = 0$*, then* $a_3 \neq 0$ *and* $F(x, y) = (a_0 - \frac{a_2^3}{a_3^2})x^3 + a_3(\frac{a_2}{a_3}x + y)^3$.

*(3) If* $D_1 \neq 0$*, then*

$$F(x, y) = \frac{\lambda_2 a_0 - D_1 a_1}{\lambda_2 - \lambda_1}(x + \frac{\lambda_1}{D_1}y)^3 + \frac{\lambda_1 a_0 - D_1 a_1}{\lambda_1 - \lambda_2}(x + \frac{\lambda_2}{D_1}y)^3, \tag{11}$$

*where* $\lambda_1$ *and* $\lambda_2$ *are the eigenvalues of the matrix* $\Lambda$ *defined by* (8).

*Proof.* Note that $F(x, y)$ is nondegenerate since $D_2^2 - 4D_1D_3 \neq 0$. We are left to prove (3). Continuing with (10), by direct computation we have

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{(\lambda_1 - \lambda_2)D_1}\begin{pmatrix} D_1 & \lambda_1 \\ -D_1 & -\lambda_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Thus we may assume $F(x, y) = \alpha(x + \frac{\lambda_1}{D_1}y)^3 + \beta(x + \frac{\lambda_2}{D_1}y)^3$ for some $\alpha$ and $\beta$. Then equation (11) follows by comparing the coefficients. ■

The following are the corresponding results for cubic equations.

**Theorem 9.** *Let* $f(x) = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$ *be a cubic polynomial with* $D_2^2 - 4D_1D_3 \neq 0$*. Then* $f(x)$ *has no multiple roots, and the roots are given as follows.*

*(1) If* $D_1 = 0$*, then* $f(x) = a_0(x + \frac{a_1}{a_0})^3 + (a_3 - \frac{a_1^3}{a_0^2})$*, and the complex roots of* $f(x)$ *are*

$$x_s = -\frac{a_1}{a_0} + \frac{\omega^s}{a_0}\sqrt[3]{a_1^3 - a_0^2a_3}, \quad s = 0, 1, 2,$$

*where* $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

*(2) If* $D_1 \neq 0$*, then* $f(x) = \frac{\lambda_2 a_0 - D_1 a_1}{\lambda_2 - \lambda_1}(x + \frac{\lambda_1}{D_1})^3 + \frac{\lambda_1 a_0 - D_1 a_1}{\lambda_1 - \lambda_2}(x + \frac{\lambda_2}{D_1})^3$*, and the complex roots of* $f(x)$ *are*

$$x_s = \frac{\gamma\omega^s\lambda_1 - \lambda_2}{D_1(1 - \gamma\omega^s)}, \quad s = 0, 1, 2, \tag{12}$$

*where* $\gamma = \sqrt[3]{\frac{\lambda_2 a_0 - D_1 a_1}{\lambda_1 a_0 - D_1 a_1}}$ *and* $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

*Proof.* We only prove (2). By (6) and (11), $f(x)$ can be completed to cubes as

$$f(x) = \frac{\lambda_2 a_0 - D_1 a_1}{\lambda_2 - \lambda_1}(x + \frac{\lambda_1}{D_1})^3 + \frac{\lambda_1 a_0 - D_1 a_1}{\lambda_1 - \lambda_2}(x + \frac{\lambda_2}{D_1})^3.$$

It is clear that

$$f(x)=0 \Leftrightarrow \frac{\lambda_2 a_0 - D_1 a_1}{\lambda_1-\lambda_2}(x+\frac{\lambda_1}{D_1})^3 = \frac{\lambda_1 a_0 - D_1 a_1}{\lambda_1-\lambda_2}(x+\frac{\lambda_2}{D_1})^3.$$

Then one readily obtains the claimed roots for the cubic equation. ■

**Theorem 10.** *Let $F(x, y) = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3$ be a cubic form. Then the following statements are equvialent.*

*(1) $Z(F) \cong \mathbb{C}[\epsilon]/(\epsilon^2)$ as algebras.*

*(2) $F(x, y)$ is nondegenerate and $D_2^2 - 4D_1D_3 = 0$.*

*(3) $F(x, y) = (\alpha_1x + \beta_1y)(\alpha_2x + \beta_2y)^2$ with $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. In this case,*

$$(\alpha_1x+\beta_1y, \alpha_2x+\beta_2y) = \begin{cases} (3a_2x+a_3y, y) & \text{if } D_1 = 0; \\ (a_0x + (3a_1 - \frac{D_2}{D_1}a_0)y, x + \frac{D_2}{2D_1}y) & \text{if } D_1 \neq 0. \end{cases}$$

*Consequently, let $f(x) = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$ be a nondegenerate cubic polynomial with $D_2^2 - 4D_1D_3 = 0$. Then $f(x)$ has a simple root and a root of multiplicity 2 given by*

$$x_0 = \frac{D_2}{D_1} - \frac{3a_1}{a_0}, \quad x_1 = x_2 = -\frac{D_2}{2D_1}.$$

*Proof.* The equivalence (1)⇔(2) follows from Proposition 6 and Theorem 7. We will show (2)⟹(3) and (3)⟹(1).

(2)⟹(3) Assume $F$ is nondegenerate and $D_2^2 - 4D_1D_3 = 0$. We make a discussion on $D_1$.

If $D_1 = 0$, then $D_2 = 0$. This implies that $\binom{a_0}{a_1} = 0$. Otherwise, $\binom{a_1}{a_2}$ and $\binom{a_2}{a_3}$ would both be multiples of $\binom{a_0}{a_1}$. This would give $D_3 = 0$, contradicting the nondegeneracy of $F$. Therefore $F(x, y) = (3a_2x + a_3y)y^2$. Moreover, $D_3 = -a_2^2 \neq 0$ implies that $a_2 \neq 0$, and hence $3a_2x + a_3y$ and $y$ are linearly independent.

Now assume $D_1 \neq 0$. Under the present condition, the matrix $\Lambda$ has two equal eigenvalues $\lambda = \lambda_{1,2} = \frac{D_2}{2}$, and the center $Z(F) \cong \mathbb{C}[\epsilon]/(\epsilon^2)$. It is clear that $\Lambda - \lambda I_2 \in Z(F)$ and $(\Lambda - \lambda I_2)^2 = 0$. By direct computation, we have

$$\begin{pmatrix} -\lambda & -D_3 \\ D_1 & \lambda \end{pmatrix}\begin{pmatrix} -\lambda & 1 \\ D_1 & 0 \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ D_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $P$ denote the invertible matrix $\begin{pmatrix} -\lambda & 1 \\ D_1 & 0 \end{pmatrix}$. Take a change of variables $\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} u \\ v \end{pmatrix}$ and denote the resulting binary cubic by $G(u, v)$. Note that $Z(G) = P^{-1}Z(F)P$. Therefore $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Z(G)$. Then by the condition of center algebras, we can easily observe that $\frac{\partial^2 G}{\partial u^2} = 0$. It follows that the monomials $u^3$ and $u^2v$ do not appear in $G(u, v)$. Therefore

$$F(x, y) = G(u, v) = b_2uv^2 + b_3v^3 = (b_2u + b_3v)v^2$$

for some $b_2, b_3 \in \mathbb{C}$. Note that $v = x + \frac{D_2}{2D_1}y$. By Vieta's formula one shows that

$$F(x, y) = [a_0x + (3a_1 - \frac{D_2}{D_1}a_0)y](x + \frac{D_2}{2D_1}y)^2.$$

(3)$\Longrightarrow$(1) Consider $G(u, v) = uv^2$. By an easy calculation,

$$Z(G) = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{C}\right\}$$

is clearly isomorphic to $\mathbb{C}[\epsilon]/(\epsilon^2)$. Now $Z(F) \cong Z(G)$ for $G$ is obtained from $F$ by taking the invertible linear change of variables $u = \alpha_1 x + \beta_1 y$, $v = \alpha_2 x + \beta_2 y$.

The last assertion follows easily from (3). Note that in this case, $a_0$ is the leading term of a cubic polynomial and hence nonzero, and $D_1 \neq 0$ as shown above. ■

Applying Theorems 9 and 10 to the cubic equation $x^3 + px + q = 0$, we obtain the well-known Cardano's formula.

**Corollary 11 (Cardano's Formula).** *Let $x^3 + px + q = 0$ be a cubic equation.*

*(1) If $\frac{p^3}{27} + \frac{q^2}{4} \neq 0$, then the equation has 3 distinct roots*

$$x_s = \omega^s \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \omega^{-s} \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \quad s = 0, 1, 2,$$

*where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and we require*

$$\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \cdot \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} = -\frac{p}{3}.$$

*(2) If $\frac{p^3}{27} + \frac{q^2}{4} = 0$ and $p \neq 0$, then the equation has 2 distinct roots*

$$x_0 = \frac{3q}{p}, \quad x_1 = x_2 = -\frac{3q}{2p}.$$

*(3) If $p = q = 0$, then the equation has roots $x_0 = x_1 = x_2 = 0$.*

*Proof.* Since (3) is obvious, we only prove (1) and (2).

Set $F(x, y) = x^3 + pxy^2 + qy^3$. By definition $D_1 = \frac{p}{3}$, $D_2 = q$, $D_3 = -\frac{p^2}{9}$, $\frac{D_2^2 - 4D_1D_3}{4} = \frac{q^2}{4} + \frac{p^3}{27}$, and $\lambda_{1,2} = \frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$. Choose cubic roots $\sqrt[3]{\lambda_1}$ and $\sqrt[3]{\lambda_2}$ such that $\sqrt[3]{\lambda_1}\sqrt[3]{\lambda_2} = -\frac{p}{3}$. Set $\sqrt[3]{-\lambda_1} = -\sqrt[3]{\lambda_1}$ and $\sqrt[3]{-\lambda_2} = -\sqrt[3]{\lambda_2}$. Then we have $\sqrt[3]{-\lambda_1}\sqrt[3]{-\lambda_2} = -\frac{p}{3}$.

(1) First assume that $\frac{p^3}{27} + \frac{q^2}{4} \neq 0$. Then $D_2^2 - 4D_1D_3 \neq 0$.

If $p \neq 0$, then $D_1 \neq 0$, and by (12) we have

$$x_s = \frac{3}{p} \cdot \frac{\lambda_1^{\frac{2}{3}}\lambda_2^{\frac{1}{3}}\omega^s - \lambda_2}{1 - \lambda_1^{-\frac{1}{3}}\lambda_2^{\frac{1}{3}}\omega^s} = \frac{3}{p}(\lambda_1\lambda_2)^{\frac{1}{3}} \frac{\lambda_1^{\frac{2}{3}}\omega^s - \lambda_2^{\frac{2}{3}}}{\lambda_1^{\frac{1}{3}} - \lambda_2^{\frac{1}{3}}\omega^s}$$

$$= \frac{3}{p}(\lambda_1\lambda_2)^{\frac{1}{3}}\omega^s \frac{\lambda_1^{\frac{2}{3}} - \lambda_2^{\frac{2}{3}}\omega^{2s}}{\lambda_1^{\frac{1}{3}} - \lambda_2^{\frac{1}{3}}\omega^s} = -(\lambda_1)^{\frac{1}{3}}\omega^s - (\lambda_2)^{\frac{1}{3}}\omega^{-s}$$

$$= \omega^s \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \omega^{-s} \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

If $p = 0$, then $D_1 = 0$. Then the equation has roots $x_s = \omega^s \sqrt[3]{-q}$, $s = 0, 1, 2$, which is the same as the stated formula.

(2) Now assume that $\frac{p^3}{27} + \frac{q^2}{4} = 0$ and $p \neq 0$. Then $D_2^2 - 4D_1D_3 = 0$ and $F(x, y)$ is nondegenerate. In fact, by Proposition 6, $F(x, y)$ is degenerate if and only $(1, 0, \frac{p}{3})$ and $(0, \frac{p}{3}, q)$ are linearly dependent, if and only if $p = q = 0$.

By Theorem 10 the equation has two distinct roots as given by

$$x_0 = \frac{3q}{p}, \quad x_1 = x_2 = -\frac{3q}{2p}. \qquad \blacksquare$$

**Remark 12.** *We have shown that the Cardano formula can be derived from our approach of completing the cube.* Moreover, the terms appearing in the Cardano formula now have some interesting meaning: they are the eigenvalues of a generating matrix of the center algebra of its associated binary cubic.

**4. SOLVING SOME ALGEBRAIC EQUATIONS BY COMPLETING POWERS.** The crux of solving cubic equations by completing the cube is that the center algebra of a general binary cubic is nontrivial, namely it contains matrices other than scalar matrices. This approach is easily extended to equations of higher degrees with nontrivial center. In particular, the key structure information of center algebras enables us to complete powers and therefore helps to find radical solutions to some algebraic equations.

For convenience, write a complex univariate polynomial of degree $d > 3$ as

$$f(x) = a_0x^d + \binom{d}{1}a_1x^{d-1} + \cdots + \binom{d}{d-1}a_{d-1}x + a_d. \tag{13}$$

We also consider its homogenization

$$F(x, y) = a_0x^d + \binom{d}{1}a_1x^{d-1}y + \cdots + \binom{d}{d-1}a_{d-1}xy^{d-1} + a_dy^d. \tag{14}$$

First of all, we compute the center of $F(x, y)$. Suppose $X = (c_{ij})_{2\times 2} \in Z(F)$. Then the condition on $Z(F)$ leads to the equation

$$F_{xx}c_{12} + F_{xy}(c_{22} - c_{11}) - F_{yy}c_{21} = 0.$$

Since

$$F_{xx} = \sum_{i=0}^{d-2} d(d-1)\binom{d-2}{i}a_ix^{d-2-i}y^i,$$

$$F_{xy} = \sum_{i=0}^{d-2} d(d-1)\binom{d-2}{i}a_{i+1}x^{d-2-i}y^i,$$

$$F_{yy} = \sum_{i=0}^{d-2} d(d-1)\binom{d-2}{i} a_{i+2} x^{d-2-i} y^i,$$

this leads to the following system of linear equations

$$\begin{cases} a_0 c_{12} + a_1(c_{22} - c_{11}) - a_2 c_{21} = 0 \\ a_1 c_{12} + a_2(c_{22} - c_{11}) - a_3 c_{21} = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{d-2} c_{12} + a_{d-1}(c_{22} - c_{11}) - a_d c_{21} = 0. \end{cases} \tag{15}$$

Since the $a_i$ are arbitrary, the rank of the coefficient matrix of the previous system of linear equations is in general 3. Thus for a general binary form $F$ of degree $> 3$, we have $c_{12} = c_{21} = 0$ and $c_{11} = c_{22}$, which leads to the conclusion that $Z(F) \cong \mathbb{C}$. This is a drastic difference in contrast to the case of $d = 3$ where the centers are automatically nontrivial as the coefficient matrix has rank at most 2. However, if $Z(F)$ is nontrivial, then $F$ can be completely determined by its center structure. Accordingly, the associated algebraic equation $f(x) = 0$ can be solved. In the following, we switch freely between $f$ and $F$ as necessary.

**Theorem 13.** *Let $f(x)$ and $F(x, y)$ be as (13) and (14).*

*(1) $f(x) = (\alpha x + \beta)^d$ if and only if* $\operatorname{rank}\begin{pmatrix} a_0 & a_1 & \cdots & a_{d-1} \\ a_1 & a_2 & \cdots & a_d \end{pmatrix} = 1$.

*(2) $f(x) = (\alpha x + \beta)^d + \gamma$ if and only if $(a_1, a_2, \dots, a_{d-1})$ is a multiple of $(a_0, a_1, \dots, a_{d-2})$.*

*(3) $f(x) = \gamma x^d + (\alpha x + \beta)^d$ if and only if $(a_1, a_2, \dots, a_{d-1})$ is a multiple of $(a_2, a_3, \dots, a_d)$.*

*(4) $f(x) = (\alpha_1 x + \beta_1)^d + (\alpha_2 x + \beta_2)^d$ with $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$ if and only if $Z(F) \cong \mathbb{C} \times \mathbb{C}$.*

*(5) $f(x) = (\alpha_1 x + \beta_1)(\alpha_2 x + \beta_2)^{d-1}$ with $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$ if and only if $Z(F) \cong \mathbb{C}[\epsilon]/(\epsilon^2)$.*

*Proof.* The proofs for the corresponding results of cubics can be applied verbatim here. To remind the reader, we also include a proof for (5) here.

For the "if" part, we simply take the change of variables $u = \alpha_1 x + \beta_1 y$ and $v = \alpha_2 x + \beta_2$, and consider $G(u, v) = uv^{d-1}$. An easy calculation shows that $Z(G) = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{C}\right\}$, and hence $Z(F) \cong Z(G) \cong \mathbb{C}[\epsilon]/(\epsilon^2)$.

For the "only if" part, we note that any nonzero nilpotent matrix $A \in \mathbb{C}^{2\times 2}$ is similar to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in P^{-1}Z(F)P$ for some invertible matrix $P$. Take the change of variables $\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} u \\ v \end{pmatrix}$. Then $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Z(G) = P^{-1}Z(F)P$, where $G(u, v)$ is the resulting binary form under the change of variables.

Assume $G(u, v) = b_0 u^d + \binom{d}{1} b_1 u^{d-1} v + \cdots + \binom{d}{d-1} b_{d-1} u v^{d-1} + b_d v^d$. Consider the linear system (15) for $G$. Then $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Z(G)$ forces $b_0 = b_1 = \cdots = b_{d-2} = 0$, that is $G(u, v) = d b_{d-1} u v^{d-1} + b_d v^d = (d b_{d-1} u + b_d v) v^{d-1}$, and the conclusion follows. ■

Finally, we derive a radical formula for an algebraic equation with nontrivial center. Keep the following notations of Section 3:

$$D_1 = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}, \; D_2 = \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}, \; D_3 = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}, \; \lambda_{1,2} = \frac{D_2 \pm \sqrt{D_2^2 - 4D_1D_3}}{2}.$$

Let $\Pi$ denote the following Hankel matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{d-2} & a_{d-1} & a_d \end{pmatrix}.$$

As the case of rank $\Pi = 1$ is easy and treated in item (1) of the previous theorem, in the rest we focus on the case of rank $\Pi = 2$.

**Theorem 14.** *Suppose* $f(x) = a_0x^d + \binom{d}{1}a_1x^{d-1} + \cdots + \binom{d}{d-1}a_{d-1}x + a_d$ *and* $a_0 \neq 0$.

*(1) Assume* rank $\Pi = 2$, $D_1 \neq 0$, *and* $D_2^2 \neq 4D_1D_3$. *Then the roots of* $f(x)$ *are*

$$x_s = \frac{\delta\zeta^s\lambda_1 - \lambda_2}{D_1(1 - \delta\zeta^s)}, \quad s = 0, 1, \ldots, d-1$$

*where* $\delta = \sqrt[d]{\frac{\lambda_2 a_0 - D_1 a_1}{\lambda_1 a_0 - D_1 a_1}}$, $\zeta = \cos\frac{2\pi}{d} + i\sin\frac{2\pi}{d}$.

*(2) Assume* rank $\Pi = 2$, $D_1 \neq 0$, *and* $D_2^2 = 4D_1D_3$. *Then the roots of* $f(x)$ *are*

$$x_0 = \frac{(d-1)D_2}{2D_1} - \frac{da_1}{a_0}, \; x_1 = \cdots = x_{d-1} = -\frac{D_2}{2D_1}.$$

*Proof.* By the assumptions rank $\Pi = 2$ and $D_1 \neq 0$, the system of linear equations (15) is determined by the first two rows. Then the center algebra $Z(F)$ is generated by

$$\Lambda = \begin{pmatrix} 0 & -D_3 \\ D_1 & D_2 \end{pmatrix}.$$

The eigenvalues of $\Lambda$ are $\lambda_{1,2} = \frac{D_2 \pm \sqrt{D_2^2 - 4D_1D_3}}{2}$.

(1) If $D_2^2 \neq 4D_1D_3$, then $\lambda_1 \neq \lambda_2$ and so $Z(F) \cong \mathbb{C} \times \mathbb{C}$. Similar to the proofs of Proposition 8 and Theorem 9, we have

$$f(x) = \frac{\lambda_2 a_0 - D_1 a_1}{\lambda_2 - \lambda_1}(x + \frac{\lambda_1}{D_1})^d + \frac{\lambda_1 a_0 - D_1 a_1}{\lambda_1 - \lambda_2}(x + \frac{\lambda_2}{D_1})^d.$$

It is clear that

$$f(x) = 0 \Leftrightarrow \frac{\lambda_2 a_0 - D_1 a_1}{\lambda_1 - \lambda_2}(x + \frac{\lambda_1}{D_1})^d = \frac{\lambda_1 a_0 - D_1 a_1}{\lambda_1 - \lambda_2}(x + \frac{\lambda_2}{D_1})^d.$$

Obviously, $f(x) = 0$ has the claimed roots.

(2) If $D_2^2 = 4D_1D_3$, then $\lambda_1 = \lambda_2 = \frac{D_2}{2}$ and thus $Z(F) \cong \mathbb{C}[\epsilon]/(\epsilon^2)$.Then similar to the proof of Theorem 10, $x = -\frac{\lambda_1}{D_1} = -\frac{D_2}{2D_1}$ is a root of multiplicity $d-1$. The other root is obtained by Vieta's formula. ■

**Remark 15.** *The condition $D_1 \neq 0$ implies that the first two columns of $\Pi$ are linearly independent. One may derive similar radical formulas when the second column and the third column are linearly independent, or the first column and the third column are linearly independent. We leave the detail to the interested reader.*

**Example 16.** *Consider the quintic equation $31x^5 + 235x^4 + 710x^3 + 1070x^2 + 805x + 242 = 0$. Then $D_1 = -8,\ D_2 = -20,\ D_3 = -12,\ \lambda_1 = -8,\ \lambda_2 = -12,\ \delta = \frac{1}{2}$. So it fits item (1) of the previous theorem and the solutions are $x_0 = -2$ and $x_n = \frac{3-e^{\frac{2\pi i}{n}}}{e^{\frac{2\pi i}{n}}-2}$ for $1 \leq n \leq 4$.*

**Example 17.** *Consider the degree 7 equation*

$$x^7 - \frac{8}{3}x^6 + \frac{11}{4}x^5 - \frac{5}{4}x^4 + \frac{5}{48}x^3 + \frac{1}{8}x^2 - \frac{3}{64}x + \frac{1}{192} = 0.$$

*Then $D_1 = -\frac{25}{1764},\ D_2 = \frac{25}{1764},\ D_3 = -\frac{25}{7056},\ \lambda_1 = \lambda_2 = \frac{25}{3528}$. So the equation fits item (2) of the previous theorem and the solutions are $x_0 = -\frac{1}{3},\ x_1 = \cdots = x_6 = \frac{1}{2}$.*

**5. SUMMARY.** In this article, we apply nontrivial center algebraic structure to provide radical solutions to some higher degree algebraic equations. In the case of cubic equations, we show that each cubic has a nontrivial center and this enables us to complete the cube, or factorize the equation. For algebraic equations of degree greater than 3, we provide very simple and elementary criteria and algorithms to complete powers and obtain radical solutions. The present method only works for very special equations, even for quartic equations. However, if we consider the completion of powers in a broader sense, then we may be able to solve more equations. In the following we take quartic equations as examples to elucidate our idea.

Let $f(x) = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4$ be a quartic. Then by a suitable change of variable, the quartic can be reduced to $g(y) = y^4 + py^2 + qy + r$. Instead of completing $g(y)$ as the sum of two biquadrates, we are content with writing $g(y)$ as a sum of two squares. Of course, this is already enough to solve the quartic equation $g(y) = 0$. We carry out the idea by the method of undetermined coefficients. Suppose

$$y^4 + py^2 + qy + r = (y^4 + 2\alpha y^2 + \alpha^2) + [(p-2\alpha)y^2 + qy + r - \alpha^2]$$

and choose $\alpha$ such that the latter quadratic term is a perfect square. This is equivalent to

$$q^2 - 4(p-2\alpha)(r-\alpha^2) = 0.$$

This is a cubic equation in $\alpha$ and clearly such an $\alpha$ is available. Now suppose

$$y^4 + py^2 + qy + r = (y^2+\alpha)^2 + (\beta y + \gamma)^2,$$

then $y^4 + py^2 + qy + r = 0$ is easily solved by simply taking the square roots of $(y^2+\alpha)^2 = -(\beta y + \gamma)^2$. Therefore quartic equations can be solved by a generalized completion of powers. It is of interest whether there is a notion of generalized centers for higher degree forms which governs such a generalization of completing the powers.

Note also that if the binary form $F(x, y)$ has center $Z(F) \cong \mathbb{C} \times \mathbb{C}$, then the splitting field of its dehomogenization $f(x)$ is usually a Kummer extension [10] over a suitable ground field. It is also of interest whether all Kummer extensions appear in this way.

**ACKNOWLEDGMENTS.** The authors are very grateful to the referees for their excellent and extraordinary reviews which help to improve the exposition significantly. This work is partially supported by the National Natural Science Foundation of China (Grant Nos. 12131015, 12161141001, 12371037 and 12371042), Fuzhou-Xiamen-Quanzhou National Independent Innovation Demonstration Zone Collaborative Innovation Platform (Grant No. 2022FX5) and the Innovation Program for Quantum Science and Technology (Grant No. 2021ZD0302902).

**DISCLOSURE STATEMENT.** No potential conflict of interest was reported by the authors.

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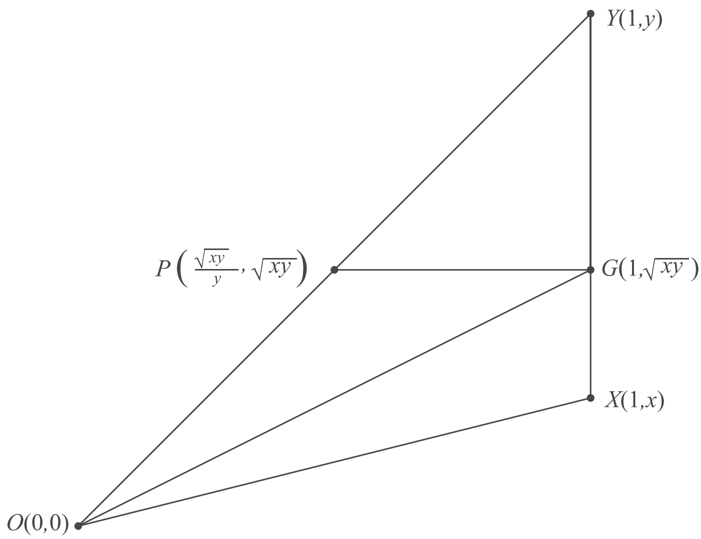
## An Arithmetic-Geometric Proof of the Arithmetic-Geometric Inequality




$$\text{Area}(OGY) > \text{Area}(OGP) = \frac{1}{2}\left(1 - \frac{\sqrt{xy}}{y}\right)\sqrt{xy} = \frac{1}{2}(\sqrt{xy} - x) \cdot 1 = \text{Area}(OXG).$$

Therefore $|GY| > |XG|$, hence $y - \sqrt{xy} > \sqrt{xy} - x$ or $\frac{x+y}{2} > \sqrt{xy}$.

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—Submitted by Alfred Witkowski, Bydgoszcz, Poland

doi.org/10.1080/00029890.2024.2433389
MSC: Primary 26E60, Secondary 97H30
This article was originally published with errors, which have now been corrected in the online version. Please see Correction (http://dx.doi.org/10.1080/00029890.2025.2455915)